Exact value of the vacuum electromagnetic energy of a dilute dielectric ball in the mode summation method

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Abstract

The exact value (in the $(\varepsilon_1 - \varepsilon_2)^2$–approximation) of the Casimir energy of a dilute dielectric ball is derived by making use of a simple and clear mode summation method. The addition theorem for the Bessel functions enables one to carry out in a closed form the summation over the angular momentum before the integration over the imaginary frequencies. The linear in $(\varepsilon_1 - \varepsilon_2)$ terms in the vacuum energy are removed by appropriate subtraction. The role of the contact terms used in other approaches to this problem is elucidated.

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I. INTRODUCTION

The progress in calculation of the Casimir energy is rather slow. In his pioneer paper [1] in 1948 H. B. G. Casimir calculated the vacuum electromagnetic energy for the most simple boundary conditions: for two parallel perfectly conducting plates placed in vacuum. When taking into account the dielectric properties of the media separated by plane boundaries, additional mathematical difficulties do not appear [2]. However the first result on the calculation of the Casimir energy for the nonflat boundaries was obtained only in 1968. By computer calculations during 3 years, T. H. Boyer found the Casimir energy of a perfectly conducting spherical shell [3]. And only just recently the vacuum electromagnetic energy of a dilute dielectric ball[1] was calculated [6–10].

The methods used for calculating the Casimir energy in the framework of the quantum field theory [11] can be classified as local or global ones. In the local approach the vacuum expectation value of the tensor of the energy–momentum density is expressed in terms of the corresponding Green's functions. Upon integrating over the space, one arrives at the Casimir energy. In the global approach, the eigenvalue of the Hamiltonian operator in the ground state is represented as the half sum of all the eigenfrequencies of the field under consideration (as the sum of zero–point energies of the relevant harmonic oscillators representing the initial quantum field).

Either of these methods has own subtle–points concerning the quantization of the electromagnetic field in media with dispersion. As known, in this case the definition of the energy–momentum tensor is not unique (the Minkowski tensor and the Abraham tensor) [12]. When calculating the Casimir energy the choice between these two possibilities should be done [13]. In the global approach one should be convinced that the imaginary part of the dielectric constant, which is for sure nonzero in real media, does not lead to absorbation of the virtual photons like the real ones. In other words, it is to be shown that the result of summing the eigenfrequencies is a real quantity. It is worth noting that it is the case in all the studies of the Casimir effect carried out.

In both approaches, the divergences appear inevitably, and unfortunately till now there are no general rules for their removing.

All the problems on the Casimir energy calculations, considered by now, were solved, as a rule, in the local and in the global approaches, the results being the same. However, for a dielectric ball it is not the case.

In calculations of the vacuum electromagnetic energy of a dilute dielectric ball conducted in papers [8–10] the Green's functions have been used essentially and an important role was played by the so called contact terms in the final expression for the vacuum energy. As it is directly followed from papers [6,8,13,14], these terms do not appear when proceeding from the sum of the eigenfrequencies because they are outside the logarithm.

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[1]The calculation of the Casimir energy in a special case, when both the material media have the same velocity of light, proves to be, from the mathematical stand point, exactly the same as for perfectly conducting shells placed in vacuum and having the shape of the interface between these media [4,5].

It is to be noted that the results of the Casimir energy calculation for a dilute dielectric ball, accomplished in the framework of the quantum field theory [8,9], coincide with those obtained by summing the van der Waals interactions between individual molecules making up the ball [6] and by applying a special perturbation theory, where the dielectric ball is treated as an perturbation in a complete Hamiltonian of electromagnetic field in unbounded empty space [7].

In this situation, the question naturally arises: how to calculate the Casimir energy of a dilute dielectric ball in the global approach or precisely how to do this by summing the frequencies of electromagnetic oscillations in the problem at hand.

This point is also important due to the following consideration. From the mathematical stand point, the most consistent method for treating the divergences in calculations of the vacuum energy is the zeta regularization technique [15]. In this method, one proceeds from the sum of the eigenfrequencies.

The analysis of the divergencies which can appear in calculation of the Casimir energy of a dielectric ball in the framework of the zeta function technique, has been carried out in [16]. It was, in particular, shown that the heat kernel coefficient a_2, responsible for the pole contribution to this Casimir energy, proves to be proportional to $(c_1-c_2)^3$ for small difference $c_1 - c_2$ (here c_1 and c_2 are the velocities of light inside and outside the ball, respectively). It implies that, in the $(c_1-c_2)^2$–approximation, the zeta renormalization technique should lead to a finite answer in this problem. However, in Ref. [16], the contact terms were certainly not considered. Therefore, without elucidating the role of these terms, the interrelation between the results of papers [8,9] and of paper [16] is not absolutely clear.

The layout of this paper is as follows. In Sect. II the derivation of the integral representation for the vacuum energy is given by making use of the mode summation and contour integration. The subtraction procedure that gives the renormalized Casimir energy in the Δn^2–approximation is discussed in detail as well as its physical justification. The addition theorem for the Bessel functions enables one to carry out the summation over the angular momentum in a closed form. It leads to an exact (in the Δn^2–approximation) value of the Casimir energy of a dilute dielectric ball. In the Conclusion (Sect. III) the method proposed here for calculating the Casimir energy is briefly discussed as well as the implication of the obtained result concerning, specifically, the elucidation of the role of the contact terms used in other approaches to this problem. In the Appendix the analysis of the divergencies in the problem at hand is fullfilled, that allows one to reveal an important relation between the linear and quadratic in Δn contributions into the vacuum energy. It is this relation that provides a simple and effective scheme of calculations which is followed in this paper.

II. MODE SUMMATION FOR VACUUM ELECTROMAGNETIC ENERGY OF A DILUTE DIELECTRIC BALL

We shall consider a solid ball of radius a placed in an unbounded uniform medium. The nonmagnetic materials making up the ball and its surroundings are characterized by permittivity ε_1 and ε_2, respectively. It is assumed that the conductivity in both the media is zero. The system of units is used where $c = \hbar = 1$.

We shall proceed from the standard definition of the vacuum energy through the eigenfrequencies of the electromagnetic oscillations [11]

$$E = \frac{1}{2} \sum_s (\omega_s - \bar{\omega}_s) \,. \tag{2.1}$$

Here ω_s are the classical frequencies of the electromagnetic field under the boundary conditions described above, and the frequencies $\bar{\omega}_s$ correspond to a certain limiting boundary conditions that will be specified below.

The sum $(1/2)\sum_s \bar{\omega}_s$ in Eq. (2.1) plays the same role as the counter terms in the standard renormalization procedure in quantum field theory [17]. However in the renormalizable field models considered in the unbounded Minkowski space–time, the explicit form of these counter terms is known (at least, it is known the algorithm of their construction at each order of perturbation theory). Unlike this, there are no general rules for obtaining the terms that should be subtracted when calculating the vacuum energy. Therefore, in any new problem on calculating the Casimir energy one needs to specify the boundary conditions, determining the frequencies $\bar{\omega}_s$, a new, appealing to some physical considerations.

Usually, it is enough to subtract in Eq. (2.1) the vacuum energy of electromagnetic field for the configuration, when the interface between two media is removed to infinity [11]. In the problem at hand it implies the limit $a \to \infty$, i.e., the medium 1 tends to fill the entire space. But it turns out that this subtraction is not sufficient because the contributions into the vacuum energy linear in $\varepsilon_1 - \varepsilon_2$ retain. Further, we assume that the difference $\varepsilon_1 - \varepsilon_2$ is small and content ourselves only with the $(\varepsilon_1 - \varepsilon_2)^2$–terms.

The necessity to subtract the contributions into the vacuum energy linear in $\varepsilon_1 - \varepsilon_2$ is justified by the following considerations. The Casimir energy of a dilute dielectric ball can be thought of as the net result of the van der Waals interactions between the molecules making up the ball [6]. These interactions are proportional to the dipole momenta of the molecules, i.e., to the quantity $(\varepsilon_1 - 1)^2$. Thus, when a dilute dielectric ball is placed in the vacuum, then its Casimir energy should be proportional to $(\varepsilon_1 - 1)^2$. It is natural to assume that when such a dielectric ball is surrounded by an infinite dielectric medium with permittivity ε_2, then its Casimir energy should be proportional to $(\varepsilon_1 - \varepsilon_2)^2$.

Further, we put for safe of symmetry

$$\sqrt{\varepsilon_1} = n_1 = 1 + \frac{\Delta n}{2} \,, \quad \sqrt{\varepsilon_2} = n_2 = 1 - \frac{\Delta n}{2} \,. \tag{2.2}$$

Here n_1 and n_1 are the refractive indices of the ball and of its surroundings, respectively, and it is assumed that $\Delta n << 1$. From here it follows, in particular, that

$$\varepsilon_1 - \varepsilon_2 = (n_1 + n_2)(n_1 - n_2) = 2\Delta n \,. \tag{2.3}$$

Thus, when using the definition (2.1) we shall keep in mind that really two subtractions should be done: first the contribution, obtained in the limit $a \to \infty$, is to be subtracted and then all the terms linear in Δn should also be removed.

We present the vacuum energy defined by Eq. (2.1) in terms of the contour integral in the complex frequency plane. The details of this procedure can be found in Refs. [4,18]. Upon the contour deformation one gets

$$E = -\frac{1}{2\pi}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty} dy\, y\, \frac{d}{dy} \ln \frac{\Delta_l^{\rm TE}(iay)\Delta_l^{\rm TM}(iay)}{\Delta_l^{\rm TE}(i\infty)\Delta_l^{\rm TM}(i\infty)}\,, \tag{2.4}$$

where $\Delta_l^{\rm TE}(iay)$ and $\Delta_l^{\rm TM}(iay)$ are the left-hand sides of the equations determining the frequencies of the electromagnetic field

$$\Delta_l^{\rm TE}(a\omega) = 0\,, \quad \Delta_l^{\rm TM}(a\omega) = 0\,. \tag{2.5}$$

For pure imaginary values of the frequency variable $\omega = iy$ (these values are needed in Eq. (2.4)), the expressions $\Delta_l^{\rm TE}(iay)$ and $\Delta_l^{\rm TM}(iay)$ are defined by

$$\begin{aligned} \Delta_l^{\rm TE}(iay) &= \sqrt{\varepsilon_1} s_l'(k_1a)e_l(k_2a) - \sqrt{\varepsilon_2} s_l(k_1a)e_l'(k_2a)\,, \\ \Delta_l^{\rm TM}(iay) &= \sqrt{\varepsilon_2} s_l'(k_1a)e_l(k_2a) - \sqrt{\varepsilon_1} s_l(k_1a)e_l'(k_2a)\,, \end{aligned} \tag{2.6}$$

where $k_i = \sqrt{\varepsilon_i}\, y$, $i = 1, 2$, and $s_l(x)$, $e_l(x)$ are the modified Riccati–Bessel functions [19]

$$s_l(x) = \sqrt{\frac{\pi x}{2}}\, I_\nu(x)\,, \quad e_l(x) = \sqrt{\frac{2x}{\pi}}\, K_\nu(x)\,, \quad \nu = l + \frac{1}{2}\,. \tag{2.7}$$

The prime in Eq. (2.6) stands for the differentiation with respect to the argument of the Riccati–Bessel functions.

The numerator (denominator) in the logarithm function in Eq. (2.4) is responsible for the first (second) term in the initial formula (2.1). For brevity we write in Eq. (2.4) simply $\Delta_l(i\infty)$ instead of $\lim_{a\to\infty}\Delta_l(iay)$. Taking into account the asymptotics of the Riccati–Bessel functions

$$s_l(x) \simeq \frac{1}{2}\, e^x\,, \quad e_l(x) \simeq e^{-x}\,, \quad x \to \infty\,,$$

we obtain

$$\Delta_l^{\rm TE}(i\infty)\Delta_l^{\rm TM}(i\infty) = -\frac{(n_1+n_2)^2}{4}\, e^{2(n_1-n_2)y}\,. \tag{2.8}$$

Upon substituting Eqs. (2.6) and (2.8) into Eq. (2.4) and changing the integration variable $ay \to y$, we cast Eq. (2.4) into the form (see Eq. (2.12) in Ref. [4])

$$\begin{aligned} E = -\frac{1}{2\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty} dy\, y\, \frac{d}{dy} \ln \Bigg\{ &\frac{4e^{-2(n_1-n_2)y}}{(n_1+n_2)^2} \\ &[n_1n_2((s_l'e_l)^2 + (s_le_l')^2) - (n_1^2+n_2^2)s_ls_l'e_le_l'] \Bigg\}\,, \end{aligned} \tag{2.9}$$

where $s_l \equiv s_l(n_1y)$, $e_l \equiv e_l(n_2y)$.

It should be noted here that in Eq. (2.9) only the first subtraction is accomplished, which removes the contribution into the vacuum energy obtained when $a \to \infty$. As noted above, for obtainng the final result all the terms linear in Δn should be subtracted from Eq. (2.9). To this end it is convenient to rewrite Eq. (2.9) in the form

$$E = E_1 + E_2 \tag{2.10}$$

with

$$E_1 = \frac{\Delta n}{2\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty} y\,dy\,, \tag{2.11}$$

$$E_2 = -\frac{1}{2\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty} dy\,y\,\frac{d}{dy}\ln\left[W_l^2(n_1y,n_2y)-\frac{\Delta n^2}{4}\,P_l^2(n_1y,n_2y)\right]\,, \tag{2.12}$$

where

$$W_l(n_1y,n_2y) = s_l(n_1y)e_l'(n_2y)-s_l'(n_1y)e_l(n_2y)\,, \tag{2.13}$$
$$P_l(n_1y,n_2y) = s_l(n_1y)e_l'(n_2y)+s_l'(n_1y)e_l(n_2y)\,. \tag{2.14}$$

The term E_1 accounts for only the expression $\exp(-2\Delta n\; y)$ in the argument of the logarithm function in Eq. (2.9).

It is worth nothing that the term E_1 is exactly the value of the Casimir energy considered by Schwinger in his attempt to explain the sonoluminescence [20]. Really, introducing the cutoff aK for frequency integration and the cutoff $y=\omega/a$ for the angular momentum summation we arrive at the result

$$E_1 = \frac{\Delta n}{\pi a}\int_0^{aK} y\,dy\sum_{l=1}^{\infty}\left(l+\frac{1}{2}\right)\sim\frac{\Delta n}{\pi a}\int_0^{aK} y^3\,dy = \Delta n\,\frac{K^4a^3}{4\pi}\,. \tag{2.15}$$

We have substituted here the summation over l by integration. Up to the multiplier $(-1/3)$ it is exactly the Schwinger value for the Casimir energy of a bubble ($\varepsilon_1=1$) in water ($\sqrt{\varepsilon_2}\simeq 4/3$) [13].

In order to obtain the renormalized vacuum energy E^{ren}, we have to subtract from Eq. (2.10) all the terms linear in Δn, i.e., E_1 and those which are generated by the function W_l^2 in E_2. How to do this, staying in the framework of the mode summation method, is shown in the Appendix.

In our calculation, we content ourselves with the Δn^2–approximation. Hence, in Eq. (2.12) one can put $P_l^2(n_1y,n_2y)\simeq P_l^2(y,y)$ and keep in expansion of the logarithm function only the terms proportional to Δn^2. In this approximation, the contributions of W_l^2 and P_l^2 into the vacuum energy are additive

$$E^{\text{ren}} = E_W + E_P\,. \tag{2.16}$$

In the Appendix it is shown that for obtaining the Δn^2–contribution into the Casimir energy of the function W_l^2 in the argument of the logarithm in Eq. (2.12), it is sufficient to calculate the Δn^2–contribution of the function W_l^2 alone changing the sign of this contribution to the opposite one. Hence,

$$E_W = \frac{1}{2\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty} dy\,y\,\frac{d}{dy}W_l^2(n_1y,n_2y)\,, \tag{2.17}$$

and only the Δn^2–term being preserved in this expression.

For E_P we have

$$E_P = \frac{\Delta n^2}{8\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty} dy\, y\frac{d}{dy}P_l^2(n_1 y, n_2 y)\,. \tag{2.18}$$

Usually, when calculating the vacuum energy in the problem with spherical symmetry, the uniform asymptotic expansion of the Bessel functions is used [19]. As a result, an approximate value of the Casimir energy can be derived, the accuracy of which depends on the number of terms preserved in the asymptotic expansion.

We shall persist in other way. By making use of the addition theorem for the Bessel functions [19], we first do the summation over the angular momentum l in Eq. (2.12) and only then we will integrate over the imaginary frequency y. As a result, we obtain an exact (in the Δn^2–approximation) value of the Casimir energy in the problem involved.

So, we further need the following addition theorem for the Bessel functions [19]

$$\sum_{l=0}^{\infty}(2l+1)s_l(\lambda r)e_l(\lambda\rho)P_l(\cos\theta) = \frac{\lambda r\rho}{R}\,e^{-\lambda R} \equiv \mathcal{D}\,, \tag{2.19}$$

where

$$R = \sqrt{r^2+\rho^2-2r\rho\cos\theta}\,. \tag{2.20}$$

Differentiating the both sides of Eq. (2.19) with respect to λr and squaring the result we deduce

$$\sum_{l=0}^{\infty}(2l+1)[s_l'(\lambda r)e_l(\lambda\rho)]^2 = \frac{1}{2r\rho}\int_{r-\rho}^{r+\rho}\left(\frac{1}{\lambda}\frac{\partial\mathcal{D}}{\partial r}\right)^2 R\,dR\,. \tag{2.21}$$

Here the orthogonality relation for the Legendre polynomials

$$\int_{-1}^{+1} P_l(x)P_m(x)dx = \frac{2\delta_{lm}}{2l+1}$$

has been used. Further we put

$$\lambda = y\,,\quad r = n_1 = 1+\frac{\Delta n}{2}\,,\quad \rho = n_2 = 1-\frac{\Delta n}{2}\,. \tag{2.22}$$

By making use of Eq. (2.21) and analogous ones, one derives

$$\begin{aligned}\sum_{l=1}^{\infty}(2l+1)W_l^2(n_1y,n_2y) &= \frac{1}{2r\rho\lambda^2}\int_{r-\rho}^{r+\rho}R\,dR\,(\mathcal{D}_r-\mathcal{D}_\rho)^2 - e^{2\Delta n y}\\ &= \frac{\Delta n^2}{8}\int_{\Delta n}^{2}\frac{e^{-2yR}}{R^5}\left(4+R^2+4yR-yR^3\right)^2 dR - e^{2\Delta n y}\,, \end{aligned}\tag{2.23}$$

$$\sum_{l=1}^{\infty}(2l+1)P_l^2(y,y) = \frac{1}{2}\int_0^2\left[\frac{\partial}{\partial y}\left(\frac{y}{R}\,e^{-yR}\right)\right]^2 R\,dR - e^{-4y}\,. \tag{2.24}$$

Here $\mathcal{D}_r$ and $\mathcal{D}_\rho$ stand for the results of the partial derivation of the function $\mathcal{D}$ in Eq. (2.19) with respect to the corresponding variables and with the subsequent substitution of (2.22). The last term in Eqs. (2.23) and (2.24) are $W_0^2(n_1y,n_2y)$ and $P_0^2(y,y)$, respectively. As it was stipulated before, we have to keep in Eq. (2.23) only the terms proportional to Δn^2 and in Eq. (2.24) we have put $\Delta n = 0$.

The calculation of the contribution E_P to the Casimir energy is straighforward. Upon differentiation of the righthanded side of Eq. (2.24) with respect to y, the integration over dR can be done here. Substitution of this result into Eq. (2.18) gives

$$E_P = -\frac{\Delta n^2}{2\pi a}\left(-\frac{1}{4}\right)\int_0^\infty dy\,\left[e^{-4y}\left(2y^2+2y-\frac{1}{2}\right)-\frac{1}{2}\right]. \tag{2.25}$$

The term $(-1/2)$ in the square brackets in Eq. (2.25) gives rise to the divergence[2] when integrating over dy, therefore we drop it with the result

$$E_P = \frac{5}{128}\,\frac{\Delta n^2}{\pi a}\,. \tag{2.26}$$

As far as the expression (2.23), it is convenient to substitute it into Eq. (2.17), do integration over y and only after that to address the integration over dR

$$\begin{aligned}\frac{\Delta n^2}{8}\int_{\Delta n}^2\int_0^\infty dy\, y\,\frac{d}{dy}\left[\frac{e^{-2yR}}{R}\left(4+R^2+4yR-yR^3\right)^2\right] &= \\ = -\frac{\Delta n^2}{4}\int_{\Delta n}^2\left(\frac{10}{R^6}+\frac{1}{R^4}+\frac{1}{8R^2}\right)dR & \\ = \frac{1}{8}\left(\frac{\Delta n^2}{3}-\frac{4}{\Delta n^3}-\frac{2}{3\Delta n}-\frac{\Delta n}{4}\right). &\end{aligned} \tag{2.27}$$

In the final result in Eq. (2.27) only the first term is to be left. The linear in Δn term should be subtracted as it was explained above (see also the Appendix). When $\Delta n \to 0$, the terms proportional to Δn^{-3} and Δn^{-1} lead to the divergencies of the same type as the $(-1/2)$ term in the square brakets in Eq. (2.25). This also concerns the term $e^{2\Delta n\, n}$ in Eq. (2.23). All these infinite contributions should be dropped. Thus, we are left with the result for the contribution E_W in Eq. (2.16)

$$E_W = \frac{1}{2\pi a}\,\frac{1}{8}\,\frac{\Delta n^2}{3} = \frac{1}{48}\,\frac{\Delta n^2}{\pi a}\,. \tag{2.28}$$

Finally we arrive at the following result for the Casimir energy of a dilute dielectric ball

$$E^{\text{ren}} = E_W + E_P = \frac{\Delta n^2}{\pi a}\left(\frac{1}{48}+\frac{5}{128}\right) = \frac{23}{384}\,\frac{\Delta n^2}{\pi a}\,. \tag{2.29}$$

Taking into account the relation (2.3) between ε_i and n_i, $i = 1, 2$, we can write

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[2]This divergence has the same origin as those arising in summation over l when the uniform asymptotic expansions of the Bessel functions are used [8,9]. The technique employed here is close to the multiple scattering expansion [21], where these divergencies are also subtracted.

$$E^{\rm ren} = \frac{23}{1536}\,\frac{(\varepsilon_1 - 1)^2}{\pi a}\,. \tag{2.30}$$

At first, this value for the Casimir energy of a dilute dielectric ball has been derived in Ref. [6] by summing the van der Waals interactions between individual molecules making up the ball. In the paper [7] this value was obtained by treating a dilute dielectric ball as a perturbation in the complete Hamiltonian of the electromagnetic field for this configuration. In papers [8,9], the result close to the exact value (2.30) has been obtained by employing the uniform asymptotic expansion of the Bessel functions.

In Ref. [4] the estimation of the Casimir energy of a dilute dielectric ball has been done taking into account, as it is seen now, only the second term in Eq. (2.29). And nevertheless it was not so bad having the accuracy about 35%.

III. CONCLUSIONS

In this paper the exact (in the Δn^2–approximation) value of the Casimir energy of a dilute dielectric ball is derived in the framework of the quantum field theory. The starting point is the mode summation by making use of the contour integration in the complex frequency plane. Unlike the other approaches to this problem, we do not use the uniform asymptotic expansion of the Bessel functions.

The key point in our consideration is employment of the addition theorem for the Bessel functions which enables us to do the summation over the angular momentum values in a closed form. As a by-product, it is shown that the role of the contact terms, at least in the Δn^2–approximation, consists only in removing the linear in Δn contributions to the Casimir energy. They do not contribute to the finite value of this energy. Remarkably, that the effect of the contact terms can be reproduced by a standard subtraction procedure, staying in the framework of the mode summation technique.

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APPENDIX: ANALYSIS OF THE DIVERGENCIES GENERATED BY W_L^2

Here we reveal an important relation between linear and quadratic in Δn terms in W_l^2 defined in Eq. (2.13).

Let us put

$$x_1 = y\left(1 + \frac{\Delta n}{2}\right), \quad x_2 = y\left(1 - \frac{\Delta n}{2}\right), \quad \Delta x = \Delta n\, y\,. \tag{A1}$$

The Taylor expansion yields

$$
\begin{aligned}
W_l(x_1,x_2) &= s_l(x_1)e_l'(x_2) - s_l'(x_1)e_l(x_2) \\
&= -1 + (2s_l'e_l' - s_le_l'' - s_l''e_l)\,\frac{\Delta x}{2} + \\
&\quad + \left[\frac{1}{2}\,(s_le_l''' - s_l'''e_l) + \frac{3}{2}\,(s_l''e_l' - s_l'e_l'')\right]\frac{\Delta x^2}{4} + O(\Delta x^3)\,.
\end{aligned} \tag{A2}
$$

For brevity we have dropped the argument y of the function s_l and e_l, and have used the value of the Wronskian

$$
W\{s_l(y), e_l(y)\} = s_le_l' - s_l'e_l = -1\,. \tag{A3}
$$

By making use of the equation for the Riccati–Bessel functions

$$
w_l''(y) - L(l,y)\,w_l(y) = 0\,, \quad L(l,y) \equiv 1 + \frac{l(l+1)}{y^2}\,, \tag{A4}
$$

we obtain

$$
\begin{aligned}
s_l'''e_l - s_le_l''' &= L(l,y)\,, \\
s_l''e_l' - s_l'e_l{\prime\prime} &= -L(l,y)\,.
\end{aligned} \tag{A5}
$$

Substitution of (A5) into (A2) gives

$$
W_l(x_1.x_2) = -1 + [s_l'e_l' - L(l,y)]\Delta x - \frac{1}{2}\,L(l,y)\,\Delta x^2 + O(\Delta x^3)\,. \tag{A6}
$$

Squaring Eq. (A6) one gets

$$
W_l^2(x_1,x_2) = 1 + A_l\,\Delta n + B_l\,\Delta n^2 + O(\Delta n^3)\,, \tag{A7}
$$

where

$$
A_l = 2y\left[2L(l,y)s_le_l - \frac{1}{2}\,(s_le_l)''\right]\,, \tag{A8}
$$

$$
B_l = y^2L(l,y) + \frac{1}{4}\,A_l^2\,. \tag{A9}
$$

Now we show that the contributions into the Casimir energy given by $\sum_l B_l$ and by $(1/4)\sum_l A_l^2$ are the same. In other words, $y^2L(l,y)$ in Eq. (A9) does not give any finite contribution into the vacuum energy. In order to prove this, we consider the following divergent expression

$$
I = \sum_{l=1}^{\infty} \nu \int_0^{\infty} y^2\,dy\,, \quad \nu = l + \frac{1}{2}\,, \tag{A10}
$$

which should be treated in the following way

$$
\begin{aligned}
I &= \lim_{s\to 0}\sum_{l=1}^{\infty}\nu\int_0^{\infty} y^{2-s}dy = \lim_{s\to 0}\sum_{l=1}^{\infty}\nu^{4-s}\int_0^{\infty} z^{2-s}dz \\
&= \lim_{s\to 0}\lim_{\mu^2\to 0}\sum_{l=1}^{\infty}\nu^{4-s}\int_0^{\infty}(z^2+\mu^2)^{1-s/2}dz\,.
\end{aligned} \tag{A11}
$$

Here the change of integration variable $y = \nu z$ is done and the photon mass μ is introduced. Further we have

$$
\begin{aligned}
I &= \lim_{s\to 0}\lim_{\mu^2\to 0}[(2^{-4+s}-1)\zeta(s-4)-2^{-4+s}]\,\frac{\mu^{3-s}}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{3}{2}+\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}-1\right)} \\
&= -\frac{\pi}{24}\lim_{s\to 0}\lim_{\mu^2\to 0}\frac{\mu^2}{\Gamma(\frac{s}{2}-1)} \to 0\,.
\end{aligned} \tag{A12}
$$

One more important inference follows from the expansion (A7). The contact terms are responsible only for the cancellation of the contributions into the vacuum energy linear in Δn, i.e. the contribution generated by A_l in Eq. (A7). This statement holds at least in the Δn^2–approximation. Really, by making use of the expansion (A7), we obtain

$$
\ln\left(W_l^2-\frac{\Delta n^2}{4}\,P_l^2\right) = A_l\Delta n + \left(B_l-\frac{A_l^2}{2}\right)\Delta n^2-\frac{\Delta n^2}{4}\,P_l^2+O(\Delta n^3)\,. \tag{A13}
$$

The terms quadratic in Δn in Eq. (A13) exactly reproduce the function $F_l(y)$ in Eq. (9) of the paper [8]. It is this function that affords the whole finite value of the Casimir energy in the problem under consideration. Thus the contact terms merely cancel the terms $A_l\Delta n$ in Eq. (A13). However, such a subtraction can also be accomplished in the framework of the mode summation method by substitution of Eq. (A13) by

$$
\ln\frac{W_l^2-(\Delta n^2/4)P_l^2}{\overline{W}_l^{\,2}} = \ln\left(W_l^2-\frac{\Delta n^2}{4}\,P_l^2\right)-\ln\overline{W}_l^{\,2}\,, \tag{A14}
$$

where $\overline{W}_l^{\,2} = 1 + A_l\Delta n$.

In view of all this, we now have the following scheme for calculating the Casimir energy in the Δn^2–approximation in the problem under consideration. First, the Δn^2–contribution should be find, which is given by the sum $\sum_l W_l^2$. Upon changing its sign to the opposite one, we obtain the contribution generated by W_l^2, when this function is in the argument of the logarithm. Obviously, this result would be deduced directly if one could find in a closed form the sum $\sum_l W_l^2 W_l^2$ [22].

Finally, we make a short comment concerning the contribution into the Casimir energy which is linear in Δn. By making use of the uniform asymptotic expansion for the Riccati–Bessel functions [14] in Eq. (A8), the sum of the A_l terms in Eq. (A13) can be represented in the form

$$
E_{\Delta n} = -\frac{1}{2\pi a}\sum_{l=1}^{\infty}\left(\frac{5\nu}{6}-\frac{7}{480\,\nu}\right)+O\left(\frac{1}{\nu^3}\right)\,. \tag{A15}
$$

This implies, in particular, that the zeta function technique does not give here a finite result in contradiction with the conclusion made in Ref. [16].

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